

# THE $(r_1, \dots, r_p)$ -BELL POLYNOMIALS

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## Abstract

In a previous paper, Mihoubi et al. introduced the  $(r_1, \dots, r_p)$ -Stirling numbers and the  $(r_1, \dots, r_p)$ -Bell polynomials and gave some of their combinatorial and algebraic properties. These numbers and polynomials generalize, respectively, the  $r$ -Stirling numbers of the second kind introduced by Broder and the  $r$ -Bell polynomials introduced by Mező. In this paper, we prove that the  $(r_1, \dots, r_p)$ -Stirling numbers of the second kind are log-concave. We also give generating functions and generalized recurrences related to the  $(r_1, \dots, r_p)$ -Bell polynomials.

**Keywords.** The  $(r_1, \dots, r_p)$ -Bell polynomials; the  $(r_1, \dots, r_p)$ -Stirling numbers; log-concavity; generalized recurrences; generating functions.

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## 1. Introduction

In 1984, Broder [2] introduced and studied the  $r$ -Stirling number of second kind  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r$ , which counts the number of partitions of the set  $[n] = \{1, 2, \dots, n\}$  into  $k$  non-empty subsets such that the  $r$  first elements are in distinct subsets. In 2011, Mező [8] introduced and studied the  $r$ -Bell polynomials. In 2012, Mihoubi et al. [11] introduced and studied the  $(r_1, \dots, r_p)$ -Stirling number of second kind  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{r_1, \dots, r_p}$ , which counts the number of partitions of the set  $[n]$  into  $k$  non-empty subsets such

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that the elements of each of the  $p$  sets

$$\begin{aligned} R_1 &:= \{1, \dots, r_1\}, \\ R_2 &:= \{r_1 + 1, \dots, r_1 + r_2\}, \\ &\dots, \\ R_p &:= \{r_1 + \dots + r_{p-1} + 1, \dots, r_1 + \dots + r_p\} \end{aligned}$$

are in distinct subsets.

This work is motivated by the study of the  $r$ -Bell polynomials [11, 8] and the  $(r_1, \dots, r_p)$ -Stirling numbers of the second kind [11], in which we may establish

- the log-concavity of the  $(r_1, \dots, r_p)$ -Stirling numbers of the second kind,
- a generalized recurrence for the  $(r_1, \dots, r_p)$ -Bell polynomials, and
- some properties for the exponential generating function of these polynomials.

To begin, by the symmetry of  $(r_1, \dots, r_p)$ -Stirling numbers respect to  $r_1, \dots, r_p$ , let us to suppose  $r_1 \leq r_2 \leq \dots \leq r_p$  and throughout this paper, we use the following notations and definitions

$$\begin{aligned} \mathbf{r}_p &:= (r_1, \dots, r_p), \quad |\mathbf{r}_p| := r_1 + \dots + r_p, \\ P_t(z; \mathbf{r}_p) &:= (z + r_p)^t (z + r_p)^{\overline{r_1}} \dots (z + r_p)^{\overline{r_{p-1}}}, \quad t \in \mathbb{R}, \\ B_n(z; \mathbf{r}_p) &:= \sum_{k=0}^{n+|\mathbf{r}_{p-1}|} \left\{ \begin{matrix} n+|\mathbf{r}_p| \\ k+r_p \end{matrix} \right\}_{\mathbf{r}_p} z^k, \quad n \geq 0 \end{aligned}$$

and  $\mathbf{e}_i$  denote the  $i$ -th vector of the canonical basis of  $\mathbb{R}^p$ .

In [11], we have proved the following

$$B_n(z; \mathbf{r}_p) = \exp(-z) \sum_{k \geq 0} P_n(k; \mathbf{r}_p) \frac{z^k}{k!} \quad (1)$$

$$P_n(z; \mathbf{r}_p) = \sum_{k=0}^{n+|\mathbf{r}_{p-1}|} \left\{ \begin{matrix} n+|\mathbf{r}_p| \\ k+r_p \end{matrix} \right\}_{\mathbf{r}_p} z^{\underline{k}}. \quad (2)$$

The following introduced numbers will be used later.

Let  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]$  be the absolute  $r$ -Stirling number of the first kind and let

$$a_k(\mathbf{r}_{p-1}) = (-1)^{|\mathbf{r}_{p-1}|-k} \sum_{|\mathbf{j}_{p-1}|=k} \left[ \begin{smallmatrix} r_1 \\ j_1 \end{smallmatrix} \right] \dots \left[ \begin{smallmatrix} r_{p-1} \\ j_{p-1} \end{smallmatrix} \right], \quad |\mathbf{j}_{p-1}| = j_1 + \dots + j_{p-1},$$

which satisfy

$$\sum_{k=0}^{|\mathbf{r}_{p-1}|} a_k(\mathbf{r}_{p-1}) u^k = (u)^{\overline{r_1}} \dots (u)^{\overline{r_{p-1}}}. \quad (3)$$

## 2. Log-concavity of the $r_p$ -Stirling numbers

In this paragraph, we discuss the real roots of the polynomial  $B_n(z; \mathbf{r}_p)$ , the log-concavity of the sequence  $\left( \left\{ \frac{n+|\mathbf{r}_p|}{k+r_p} \right\}_{\mathbf{r}_p}, 0 \leq k \leq n+|\mathbf{r}_{p-1}| \right)$ , the greatest maximizing index of  $\left\{ \frac{n}{k} \right\}_{\mathbf{r}_p}$  and we give an approximation of  $\left\{ \frac{n+|\mathbf{r}_p|}{m+r_p} \right\}_{\mathbf{r}_p}$  when  $n$  tends to infinity. The case  $p = 1$  was studied by Mező [9] and other study is given by Zhao [14] on a large class of Stirling numbers.

**Theorem 1.** *The roots of the polynomial  $B_n(z; \mathbf{r}_p)$  are real, distinct and negative.*

*Proof.* We have

$$\begin{aligned}
 & \frac{d^{r_{p+1}}}{dz^{r_{p+1}}} (z^{r_p} \exp(z) B_n(z; \mathbf{r}_p)) \\
 &= \frac{d^{r_{p+1}}}{dz^{r_{p+1}}} \left( \sum_{k \geq 0} (k+r_p)^{\overline{r_1}} \cdots (k+r_p)^{\overline{r_p}} (k+r_p)^n \frac{z^{k+r_p}}{(k+r_p)!} \right) \\
 &= \sum_{k \geq r_{p+1}-r_p} (k+r_p)^{\overline{r_1}} \cdots (k+r_p)^{\overline{r_p}} (k+r_p)^n \frac{z^{k+r_p-r_{p+1}}}{(k+r_p-r_{p+1})!} \\
 &= \sum_{k \geq 0} (k+r_{p+1})^{\overline{r_1}} \cdots (k+r_{p+1})^{\overline{r_p}} (k+r_{p+1})^n \frac{z^k}{k!}, \\
 &= \exp(z) B_n(z; \mathbf{r}_{p+1}).
 \end{aligned}$$

This gives

$$B_n(z; \mathbf{r}_{p+1}) = \exp(-z) \frac{d^{r_{p+1}}}{dz^{r_{p+1}}} (z^{r_p} \exp(z) B_n(z; \mathbf{r}_p)). \quad (4)$$

Now, we show by induction that the roots of the polynomials  $B_n(z; \mathbf{r}_p)$  are distinct and negative reals.

For  $\mathbf{r}_p = \mathbf{0}$ , the polynomial  $B_n(z; \mathbf{r}_p)$  being the classical Bell polynomial and for  $\mathbf{r}_{p-1} = \mathbf{0}$ ,  $r_p = r$ , it being the  $r$ -Bell polynomial introduced in [8] which it is known that these polynomials have only distinct negative real roots.

Assume that  $B_n(z; \mathbf{r}_j)$  have only distinct negative real roots for  $1 \leq j \leq p$ .

Because the polynomials  $\exp(-z) \frac{d^j}{dz^j} (z^{r_p} \exp(z) B_n(z; \mathbf{r}_p))$ ,  $0 \leq j \leq r_{p+1}$ , are of the same degree  $n+|\mathbf{r}_p|$ , we conclude that the polynomial  $z^{r_p} B_n(z; \mathbf{r}_p)$  has  $n+|\mathbf{r}_{p-1}|$  distinct negative real roots and  $z=0$  is a root of multiplicity  $r_p$ . Then, the Rolle's theorem shows that the polynomial  $\exp(-z) \frac{d}{dz} (z^{r_p} \exp(z) B_n(z; \mathbf{r}_p))$  has  $n+|\mathbf{r}_{p-1}|$  distinct negative real roots and  $z=0$  is a root of multiplicity  $r_p-1$ , and because the polynomial  $z^{-(r_p-1)} \exp(-z) \frac{d}{dz} (z^{r_p} \exp(z) B_n(z; \mathbf{r}_p))$  is of degree

$n + |\mathbf{r}_{p-1}| + 1$ , then all its roots are distinct and real negative. We proceed similarly for the successive derivatives to obtain the same property for the polynomial  $\exp(-z) \frac{d^{r_p+1}}{dz^{r_p+1}} (z^{r_p} \exp(z) B_n(z; \mathbf{r}_p)) = B_n(z; \mathbf{r}_{p+1})$ , see (4).  $\square$

On using Newton's inequality given in Hardy, Littlewood and Polya [6, p. 52] we may state that:

**Corollary 2.** *The sequence  $\left\{ \left\{ \begin{smallmatrix} n+|\mathbf{r}_p| \\ k+r_p \end{smallmatrix} \right\}_{\mathbf{r}_p}, 0 \leq k \leq n + |\mathbf{r}_{p-1}| \right\}$  is log-concave.*

This property shows that the sequence  $(\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\mathbf{r}_p}, 0 \leq k \leq n)$  admits an index  $K \in \{0, 1, \dots, n\}$  for which  $\left\{ \begin{smallmatrix} n \\ K \end{smallmatrix} \right\}_{\mathbf{r}_p}$  being the maximum of  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\mathbf{r}_p}$ . The following corollary gives a small interval for this index.

**Corollary 3.** *Let  $K_{n, \mathbf{r}_p}$  be the greatest maximizing index of  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\mathbf{r}_p}$ . We have*

$$\left| K_{n+|\mathbf{r}_p|, \mathbf{r}_p} - \left( \frac{B_{n+1}(1; \mathbf{r}_p)}{B_n(1; \mathbf{r}_p)} - (r_p + 1) \right) \right| < 1.$$

*Proof.* Since the sequence  $\left\{ \begin{smallmatrix} n+|\mathbf{r}_p| \\ k+r_p \end{smallmatrix} \right\}_{\mathbf{r}_p}$  is log-concave, there exists an index  $K_{n+|\mathbf{r}_p|, \mathbf{r}_p}$  for which

$$\left\{ \begin{smallmatrix} n+|\mathbf{r}_p| \\ r_p \end{smallmatrix} \right\}_{\mathbf{r}_p} < \dots < \left\{ \begin{smallmatrix} n+|\mathbf{r}_p| \\ K_{n+|\mathbf{r}_p|, \mathbf{r}_p} \end{smallmatrix} \right\}_{\mathbf{r}_p} > \dots > \left\{ \begin{smallmatrix} n+|\mathbf{r}_p| \\ n+r_p \end{smallmatrix} \right\}_{\mathbf{r}_p}.$$

Then, on applying Theorem 1 and the Darroch's theorem [3] (see also [9]), we obtain

$$\left| K_{n+|\mathbf{r}_p|, \mathbf{r}_p} - \frac{\frac{d}{dz} B_n(z; \mathbf{r}_p)|_{z=1}}{B_n(1; \mathbf{r}_p)} \right| < 1.$$

It remains to apply Corollary 14 given in [11] and use the identity

$$z \frac{d}{dz} (B_n(z; \mathbf{r}_p)) = B_{n+1}(z; \mathbf{r}_p) - (z + r_p) B_n(z; \mathbf{r}_p).$$

$\square$

### 3. Generalized recurrences and generating functions

The polynomial  $B_n(z; \mathbf{r}_p)$  admits in the basis  $\{B_{n+k}(z; r_p) : 0 \leq k \leq n + |\mathbf{r}_{p-1}|\}$  the following representation:

**Theorem 4.** *We have*

$$\begin{aligned} B_n(z; \mathbf{r}_p) &= \sum_{k=0}^{|\mathbf{r}_{p-1}|} a_k(\mathbf{r}_{p-1}) B_{n+k}(z; r_p), \\ B_n(z; \mathbf{r}_{p+q}) &= \sum_{k=0}^{|\mathbf{r}_{p-1}|} a_k(\mathbf{r}_{p-1}) B_{n+k}(z; r_p, \dots, r_{p+q}). \end{aligned}$$

*Proof.* On the using the fact that

$$(k + r_p)^{\overline{r_m}} = \sum_{j=0}^{r_m} (-1)^{n-j} \begin{bmatrix} r_m \\ j \end{bmatrix} (k + r_p)^j,$$

we get

$$\begin{aligned} B_n(z; \mathbf{r}_p) &= \exp(-z) \sum_{k \geq 0} P_n(k; \mathbf{r}_p) \frac{z^k}{k!} \\ &= \exp(-z) \sum_{k \geq 0} \sum_{j=0}^{r_m} (-1)^{r_m-j} \begin{bmatrix} r_m \\ j \end{bmatrix} \frac{P_0(k; \mathbf{r}_p)}{(k + r_p)^{\overline{r_m}}} (k + r_p)^{n+j} \frac{z^k}{k!} \\ &= \sum_{j=0}^{r_m} (-1)^{r_m-j} \begin{bmatrix} r_m \\ j \end{bmatrix} B_{n+j}(z; \mathbf{r}_p - r_m \mathbf{e}_m), \quad m = 1, 2, \dots, p-1. \end{aligned}$$

with the process, we obtain

$$\begin{aligned} B_n(z; \mathbf{r}_p) &= \sum_{j_1=0}^{r_1} \cdots \sum_{j_{p-1}=0}^{r_{p-1}} (-1)^{|\mathbf{r}_{p-1}| - |\mathbf{j}_{p-1}|} \begin{bmatrix} r_1 \\ j_1 \end{bmatrix} \cdots \begin{bmatrix} r_{p-1} \\ j_{p-1} \end{bmatrix} B_{n+|\mathbf{j}_{p-1}|}(z; r_p) \\ &= \sum_{k=0}^{|\mathbf{r}_{p-1}|} (-1)^{|\mathbf{r}_{p-1}| - k} B_{n+k}(z; r_p) \sum_{|\mathbf{j}_{p-1}|=k} \begin{bmatrix} r_1 \\ j_1 \end{bmatrix} \cdots \begin{bmatrix} r_{p-1} \\ j_{p-1} \end{bmatrix} \\ &= \sum_{k=0}^{|\mathbf{r}_{p-1}|} a_k(\mathbf{r}_{p-1}) B_{n+k}(z; r_p). \end{aligned}$$

This implies the first identity of the theorem.

The second identity of the theorem can be obtained on utilizing identity (4)  $q$  times on the two sides of the first identity of the theorem.  $\square$

**Corollary 5.** *We have*

$$\left\{ \begin{matrix} n + |\mathbf{r}_p| \\ k + r_p \end{matrix} \right\}_{\mathbf{r}_p} = \sum_{j=0}^{|\mathbf{r}_{p-1}|} \left\{ \begin{matrix} n + j + r_p \\ k + r_p \end{matrix} \right\}_{\mathbf{r}_p} a_j(\mathbf{r}_{p-1}).$$

*Proof.* From Theorem 4 we obtain

$$\begin{aligned}
B_n(z; \mathbf{r}_p) &= \sum_{j=0}^{|\mathbf{r}_{p-1}|} a_j(\mathbf{r}_{p-1}) B_{n+j}(z; r_p) \\
&= \sum_{j=0}^{|\mathbf{r}_{p-1}|} a_j(\mathbf{r}_{p-1}) \sum_{k=0}^{n+j} \left\{ \begin{matrix} n+j+r_p \\ k+r_p \end{matrix} \right\}_{\mathbf{r}_p} z^k \\
&= \sum_{k=0}^{n+|\mathbf{r}_{p-1}|} z^k \sum_{j=0}^{|\mathbf{r}_{p-1}|} a_j(\mathbf{r}_{p-1}) \left\{ \begin{matrix} n+j+r_p \\ k+r_p \end{matrix} \right\}_{\mathbf{r}_p}
\end{aligned}$$

and because  $B_n(z; \mathbf{r}_p) = \sum_{k=0}^{n+|\mathbf{r}_{p-1}|} \left\{ \begin{matrix} n+|\mathbf{r}_p| \\ k+r_p \end{matrix} \right\}_{\mathbf{r}_p} z^k$ , the identity follows by identification.  $\square$

In [11], we have proved the following

$$\sum_{n \geq 0} B_n(z; \mathbf{r}_p) \frac{t^n}{n!} = B_0(z \exp(t); \mathbf{r}_p) \exp(z(\exp(t) - 1) + r_p t).$$

The following theorem gives more details on the exponential generating function of the  $\mathbf{r}_p$ -Bell polynomials.

**Theorem 6.** *We have*

$$\begin{aligned}
\sum_{n \geq 0} B_{n+m}(z; \mathbf{r}_p) \frac{t^n}{n!} &= B_m(z \exp(t); \mathbf{r}_p) \exp(z(\exp(t) - 1) + r_p t) \\
&= \sum_{k=0}^{|\mathbf{r}_{p-1}|} a_k(\mathbf{r}_{p-1}) \frac{d^{m+k}}{dt^{m+k}} (\exp(z(\exp(t) - 1) + r_p t)).
\end{aligned}$$

*Proof.* Use (1) to get

$$\begin{aligned}
\sum_{n \geq 0} B_{n+m}(z; \mathbf{r}_p) \frac{t^n}{n!} &= \sum_{n \geq 0} B_n(z; \mathbf{r}_p) \frac{t^n}{n!} \\
&= \sum_{n \geq 0} \left( \exp(-z) \sum_{k \geq 0} P_0(k; \mathbf{r}_p) (k + r_p)^{n+m} \frac{z^k}{k!} \right) \frac{t^n}{n!} \\
&= \exp(-z) \sum_{k \geq 0} P_0(k; \mathbf{r}_p) (k + r_p)^m \frac{z^k \exp((k + r_p)t)}{k!} \\
&= B_m(z \exp(t); \mathbf{r}_p) \exp(z(\exp(t) - 1) + r_p t).
\end{aligned}$$

For the second part of the theorem, use Theorem 4 to obtain

$$\begin{aligned}
\sum_{n \geq 0} B_{n+m}(z; \mathbf{r}_p) \frac{t^n}{n!} &= \sum_{k=0}^{|\mathbf{r}_{p-1}|} a_k(\mathbf{r}_{p-1}) \sum_{n \geq 0} B_{n+m+k}(z; r_p) \frac{t^n}{n!} \\
&= \sum_{k=0}^{|\mathbf{r}_{p-1}|} a_k(\mathbf{r}_{p-1}) \frac{d^{m+k}}{dt^{m+k}} \left( \sum_{n \geq 0} B_n(z; r_p) \frac{t^n}{n!} \right) \\
&= \sum_{k=0}^{|\mathbf{r}_{p-1}|} a_k(\mathbf{r}_{p-1}) \frac{d^{m+k}}{dt^{m+k}} (\exp(z(\exp(t) - 1) + r_p t)).
\end{aligned}$$

□

Spivey [12] gave a beautiful combinatorial identity, after that, in different ways, Belbachir et al. [1], Gould et al. [7], generalized this identity on showing that the polynomial  $B_{n+m}(z) = B_{n+m}(z; \mathbf{0})$  admits a recurrence relation related to the family of  $\{z^i B_j(z)\}$  as follows

$$B_{n+m}(z) = \sum_{k=0}^n \sum_{j=0}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \binom{n}{k} j^{n-k} z^j B_k(z), \quad (5)$$

and recently, Xu [13] generalized these results on giving recurrence relation on a large family on Stirling numbers. Other recurrence relations are given by Mező [10]. The following theorem generalizes identity (5), the Carlitz's identities [4, 5] given by

$$\begin{aligned}
B_{n+m}(1; r) &= \sum_{k=0}^m \left\{ \begin{matrix} m+r \\ k+r \end{matrix} \right\}_r B_n(1; k+r) \quad \text{and} \\
B_n(1; r+s) &= \sum_{k=0}^s \left[ \begin{matrix} s+r \\ k+r \end{matrix} \right]_r (-1)^{s-k} B_{n+k}(1; r)
\end{aligned}$$

and shows that  $B_{n+m}(z; \mathbf{r}_p)$  admits  $r$ -Stirling recurrence coefficients in the families of basis

$$\begin{aligned}
&\{z^j B_m(z; \mathbf{r}_p + j\mathbf{e}_p) : 0 \leq j \leq n\} \quad \text{and} \\
&\{z^j B_{m+i}(z; r+j) : 0 \leq i \leq |\mathbf{r}_{p-1}|, 0 \leq j \leq n\},
\end{aligned}$$

where  $B_n(1; r)$  is the number of ways to partition a set of  $n$  elements into non-empty subsets such that the  $r$  first elements are in different subsets.

**Theorem 7.** *We have*

$$\begin{aligned} B_{n+m}(z; \mathbf{r}_p) &= \sum_{j=0}^n \left\{ \begin{matrix} n+r \\ j+r_p \end{matrix} \right\}_{r_p} z^j B_m(z; \mathbf{r}_p + j\mathbf{e}_p), \\ B_{n+m}(z; \mathbf{r}_p) &= \sum_{i=0}^{|\mathbf{r}_{p-1}|} \sum_{j=0}^n \left\{ \begin{matrix} n+r \\ j+r_p \end{matrix} \right\}_{r_p} a_i(\mathbf{r}_{p-1}) z^j B_{m+i}(z; \mathbf{r}_p + j), \\ z^n B_m(z; \mathbf{r}_p + n\mathbf{e}_p) &= \sum_{j=0}^n \left[ \begin{matrix} n+r_p \\ j+r_p \end{matrix} \right]_{r_p} (-1)^{n-j} B_{m+j}(z; \mathbf{r}_p). \end{aligned}$$

*Proof.* Let

$$T_m(z; \mathbf{r}_p) := \sum_{n \geq 0} \left( \sum_{j=0}^n \left\{ \begin{matrix} n+r \\ j+r_p \end{matrix} \right\}_{r_p} z^j B_m(z; \mathbf{r}_p + j\mathbf{e}_p) \right) \frac{t^n}{n!}.$$

We use Corollary 12 given in [11] to verify that we have

$$B_m(z; \mathbf{r}_p + j\mathbf{e}_p) = \exp(-z) \frac{d^j}{dz^j} (\exp(z) B_m(z; \mathbf{r}_p)). \quad (6)$$

Identity (6) and the following generating function (see [2])

$$\sum_{n \geq j} \left\{ \begin{matrix} n+r \\ j+r_p \end{matrix} \right\}_{r_p} \frac{t^n}{n!} = \frac{1}{j!} (\exp(t) - 1)^j \exp(r_p t)$$

prove that

$$\begin{aligned} T_m(z; \mathbf{r}_p) &= \sum_{j \geq 0} B_m(z; \mathbf{r}_p + j\mathbf{e}_p) z^j \frac{1}{j!} (\exp(t) - 1)^j \exp(r_p t) \\ &= \exp(r_p t - z) \sum_{j \geq 0} \frac{d^j}{dz^j} (\exp(z) B_m(z; \mathbf{r}_p)) \frac{(z(\exp(t) - 1))^j}{j!}. \end{aligned}$$

Now, by the Taylor-Maclaurin's expansion we have

$$\sum_{j \geq 0} \frac{d^j}{dz^j} (\exp(z) B_m(z; \mathbf{r}_p)) \frac{(u - z)^j}{j!} = \exp(u) B_m(u; \mathbf{r}_p),$$

and we get

$$T_m(z; \mathbf{r}_p) = \exp(r_p t - z) \exp(z \exp(t)) B_m(z \exp(t); \mathbf{r}_p).$$

By Theorem 6, we obtain

$$T_m(z; \mathbf{r}_p) = \sum_{n \geq 0} B_{n+m}(z; \mathbf{r}_p) \frac{t^n}{n!}$$



By identification, the first identity of Theorem follows.

The second identity follows on utilizing Theorem 4 to replace  $B_m(z; \mathbf{r}_p + j\mathbf{e}_p)$  by

$$B_m(z; \mathbf{r}_p + j\mathbf{e}_p) = \sum_{i=0}^{|\mathbf{r}_{p-1}|} a_i(\mathbf{r}_{p-1}) B_{m+i}(z; j + r_p).$$

For the third identity, we use the identity (1) and  $(k + r_p)^{\underline{n}} = \sum_{j=0}^n \begin{bmatrix} n+r_p \\ j+r_p \end{bmatrix}_{r_p} k^j$ , see [2], to obtain

$$\begin{aligned} & \sum_{j=0}^n \begin{bmatrix} n+r_p \\ j+r_p \end{bmatrix}_{r_p} (-1)^{n-j} B_{m+j}(z; \mathbf{r}_p) \\ &= \exp(-z) \sum_{k \geq 0} P_m(k; \mathbf{r}_p) \frac{z^k}{k!} \sum_{j=0}^n \begin{bmatrix} n+r_p \\ j+r_p \end{bmatrix}_{r_p} (-1)^{n-j} (k + r_p)^j \\ &= (-1)^n \exp(-z) \sum_{k \geq 0} P_m(k; \mathbf{r}_p) \frac{z^k}{k!} \sum_{j=0}^n \begin{bmatrix} n+r_p \\ j+r_p \end{bmatrix}_{r_p} (-k - r_p)^j \\ &= (-1)^n \exp(-z) \sum_{k \geq 0} P_m(k; \mathbf{r}_p) \frac{z^k}{k!} (-k - r_p + r_p)^{\overline{n}} \\ &= \exp(-z) \sum_{k \geq n} P_m(k; \mathbf{r}_p) k^{\underline{n}} \frac{z^k}{k!} \\ &= z^n \exp(-z) \sum_{k \geq 0} P_m(k + n; \mathbf{r}_p) \frac{z^k}{k!} \\ &= z^n B_m(z; \mathbf{r}_p + n\mathbf{e}_p). \end{aligned}$$

□

**Corollary 8.** *We have*

$$\begin{aligned} & \sum_{i=0}^k \left\{ \begin{matrix} m + |\mathbf{r}_p| \\ i + r_p \end{matrix} \right\}_{\mathbf{r}_p} \left\{ \begin{matrix} n + r_p \\ k - i + r_p \end{matrix} \right\}_{r_p} = \left\{ \begin{matrix} n + m + |\mathbf{r}_p| \\ k + r_p \end{matrix} \right\}_{\mathbf{r}_p}, \\ & \sum_{j=0}^n \left\{ \begin{matrix} m + j + |\mathbf{r}_p| \\ k + n + r_p \end{matrix} \right\}_{\mathbf{r}_p} \begin{bmatrix} n + r_p \\ j + r_p \end{bmatrix}_{r_p} (-1)^{n-j} = \left\{ \begin{matrix} m + |\mathbf{r}_p| + n \\ k + r_p + n \end{matrix} \right\}_{\mathbf{r}_p + n\mathbf{e}_p}, \\ & \sum_{j=0}^n \left\{ \begin{matrix} m + j + |\mathbf{r}_p| \\ k + r_p \end{matrix} \right\}_{\mathbf{r}_p} \begin{bmatrix} n + r_p \\ j + r_p \end{bmatrix}_{r_p} (-1)^{n-j} = 0, \quad k < n. \end{aligned}$$

*Proof.* From the second identity of Theorem 7 we obtain

$$\begin{aligned}
B_{n+m}(z; \mathbf{r}_p) &= \sum_{j=0}^n \left\{ \begin{matrix} n+r_p \\ j+r_p \end{matrix} \right\}_{r_p} z^j B_m(z; \mathbf{r}_p + j\mathbf{e}_p) \\
&= \sum_{j=0}^n \left\{ \begin{matrix} n+r_p \\ j+r_p \end{matrix} \right\}_{r_p} z^j \sum_{i=0}^{m+|\mathbf{r}_{p-1}|} \left\{ \begin{matrix} m+|\mathbf{r}_p| \\ i+r_p \end{matrix} \right\}_{\mathbf{r}_p} z^i \\
&= \sum_{k=0}^{n+m+|\mathbf{r}_{p-1}|} z^k \sum_{i=0}^k \left\{ \begin{matrix} m+|\mathbf{r}_p| \\ i+r_p \end{matrix} \right\}_{\mathbf{r}_p} \left\{ \begin{matrix} n+r_p \\ k-i+r_p \end{matrix} \right\}_{r_p}.
\end{aligned}$$

Then, use the definition of  $B_{n+m}(z; \mathbf{r}_p)$ , the desired identity follows by identification.

On using the definition of  $B_n(z; \mathbf{r}_p)$  and the third identity of Theorem 7, the second and the third identities of the corollary follow from the following expansion:

$$\begin{aligned}
&\sum_{k=0}^{m+|\mathbf{r}_{p-1}|} \left\{ \begin{matrix} m+|\mathbf{r}_p|+n \\ k+r_p+n \end{matrix} \right\}_{\mathbf{r}_p+n\mathbf{e}_p} z^k \\
&= B_m(z; \mathbf{r}_p + n\mathbf{e}_p) \\
&= z^{-n} \sum_{j=0}^n \left[ \begin{matrix} n+r_p \\ j+r_p \end{matrix} \right]_{r_p} (-1)^{n-j} B_{m+j}(z; \mathbf{r}_p) \\
&= z^{-n} \sum_{j=0}^n \left[ \begin{matrix} n+r_p \\ j+r_p \end{matrix} \right]_{r_p} (-1)^{n-j} \sum_{k=0}^{m+j+|\mathbf{r}_{p-1}|} \left\{ \begin{matrix} m+j+|\mathbf{r}_p| \\ k+r_p \end{matrix} \right\}_{\mathbf{r}_p} z^k \\
&= \sum_{k=0}^{m+n+|\mathbf{r}_{p-1}|} z^{k-n} \sum_{j=0}^n \left\{ \begin{matrix} m+j+|\mathbf{r}_p| \\ k+r_p \end{matrix} \right\}_{\mathbf{r}_p} \left[ \begin{matrix} n+r_p \\ j+r_p \end{matrix} \right]_{r_p} (-1)^{n-j} \\
&= \sum_{k=-n}^{m+|\mathbf{r}_{p-1}|} z^k \sum_{j=0}^n \left\{ \begin{matrix} m+j+|\mathbf{r}_p| \\ k+n+r_p \end{matrix} \right\}_{\mathbf{r}_p} \left[ \begin{matrix} n+r_p \\ j+r_p \end{matrix} \right]_{r_p} (-1)^{n-j}.
\end{aligned}$$

□

The *o.g.f.* of the  $r$ -Stirling numbers of the second kind [2] given by

$$\sum_{n \geq k} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r t^n = t^k \prod_{j=0}^k (1 - (r+j)t)^{-1}, \quad (7)$$

An analogue result for the  $\mathbf{r}_p$ -Stirling numbers is given by the following theorem.

**Theorem 9.** *Let*

$$\tilde{B}_n(z; \mathbf{r}_p) := \sum_{k=0}^n \left\{ \begin{matrix} n+|\mathbf{r}_p| \\ k+|\mathbf{r}_p| \end{matrix} \right\}_{\mathbf{r}_p} z^k.$$

Then, we have

$$\begin{aligned} \sum_{n \geq k} \left\{ \begin{matrix} n + |\mathbf{r}_p| \\ k + |\mathbf{r}_p| \end{matrix} \right\}_{\mathbf{r}_p} t^n &= t^{k+|\mathbf{r}_{p-1}|} \left( \frac{1}{t} \right)^{r_1} \cdots \left( \frac{1}{t} \right)^{r_{p-1}} \prod_{j=0}^{k+|\mathbf{r}_{p-1}|} (1 - (r_p + j)t)^{-1}, \\ \sum_{n \geq 0} \tilde{B}_n(z; \mathbf{r}_p) t^n &= \left( \frac{1}{t} \right)^{r_1} \cdots \left( \frac{1}{t} \right)^{r_{p-1}} \sum_{k \geq |\mathbf{r}_{p-1}|} \frac{z^{k-|\mathbf{r}_{p-1}|} t^k}{\prod_{j=0}^k (1 - (r_p + j)t)}. \end{aligned}$$

*Proof.* Use Corollary 5 to obtain

$$\begin{aligned} \sum_{n \geq k} \left\{ \begin{matrix} n + |\mathbf{r}_p| \\ k + |\mathbf{r}_p| \end{matrix} \right\}_{\mathbf{r}_p} t^n &= \sum_{n \geq k} \left\{ \begin{matrix} n + |\mathbf{r}_p| \\ k + |\mathbf{r}_{p-1}| + r_p \end{matrix} \right\}_{\mathbf{r}_p} t^n \\ &= \sum_{j=0}^{|\mathbf{r}_{p-1}|} a_j(\mathbf{r}_{p-1}) t^{-j} \sum_{n \geq k} \left\{ \begin{matrix} n + j + r_p \\ k + |\mathbf{r}_{p-1}| + r_p \end{matrix} \right\}_{\mathbf{r}_p} t^{n+j} \\ &= \left( \sum_{n \geq k+|\mathbf{r}_{p-1}|} \left\{ \begin{matrix} n + r_p \\ k + |\mathbf{r}_{p-1}| + r_p \end{matrix} \right\}_{\mathbf{r}_p} t^n \right) \left( \sum_{j=0}^{|\mathbf{r}_{p-1}|} a_j(\mathbf{r}_{p-1}) t^{-j} \right). \end{aligned}$$

The first generating function of the theorem follows by using (3) and the generating function given by 7. For the second one, use the definition of  $B_n(z; \mathbf{r}_p)$  and the last expansion.  $\square$

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